ASYMPTOTIC FIELDS AT COHESIVE CRACK TIPS

B L Karihaloo
School of Engineering, Cardiff University, Cardiff CF24 3AA, U.K.

The stress and displacement fields at the front of propagating fracture which depend on the actual loading on the structure and its boundary conditions are needed to study its growth. In this paper we shall review these fields for cohesive cracks by which are meant a traction-free fracture front with a large process zone ahead of it in which the material experiences progressive softening (Fig. 1). Over the process (or cohesive) zone the material is able to transfer some traction across the crack faces depending upon how much the faces have separated or slid relative to each other (Karihaloo, 1995) which are prescribed by cohesion-separation relationships. The cohesive cracks under consideration are therefore different from both Barenblatt (1959), Dugdale (1960) and BCS (1963) simplified models.

The lack of any work on the asymptotic fields at the tips of cohesive cracks belies the widespread use of cohesive crack models. The solution of asymptotic fields at cohesive crack tips was obtained very recently by Xiao & Karihaloo (2006). They considered only normal cohesion-separation relationships, but allowed for the effect of Coulomb friction on the cohesive crack faces. The special case of a pure mode I cohesive crack was fully investigated. Their solution is valid for any cohesion-separation law that can be expressed in a special polynomial form. They showed that many commonly used separation laws, e.g. rectangular, linear, bilinear, exponential, etc can be easily expressed in this form. They used these asymptotic fields as enrichment functions in the extended/generalized finite element method at the tip of long cohesive cracks, as well as short branches/kinks. These asymptotic crack tip fields parallel those of Williams (1957) for brittle materials.

Asymptotic Fields at Frictionless and Frictional Cohesive Crack Tips

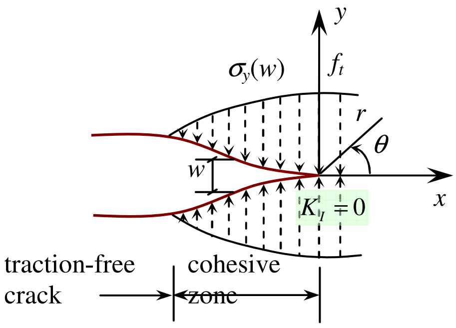

Fig 1. A real traction-free crack terminating in a fracture process (cohesive) zone (FPZ) with residual stress transfer capacity $\sigma_y(w)$ whose faces close smoothly near its tip ($K_I = 0$). The material outside the FPZ is linear elastic, but within the FPZ is softening.

In order to simplify the derivation of the cohesive crack tip asymptotic fields, we will represent cohesion-separation law by the following general polynomial

$$\frac{\sigma}{f_t} = 1 + \sum_{i=1}^{5} \alpha_i \left(\frac{w}{w_c}\right)^{\frac{2}{3}i} - \left(1 + \sum_{i=1}^{5} \alpha_i\right)\left(\frac{w}{w_c}\right)^4 \qquad (1)$$

where α_i, $i = 1 \sim 5$, are fitting parameters. Xiao & Karihaloo (2006) showed that the commonly used exponential (2), linear (3), bilinear (4) and rectangular (5) cohesion-separation laws given below can be faithfully expressed in the general form (1).

$$\frac{\sigma}{f_t} = f\left(\frac{w}{w_c}\right) - \frac{w}{w_c} f(1); \qquad f\left(\frac{w}{w_c}\right) = \left[1 + \left(C_1 \frac{w}{w_c}\right)^3\right] e^{-C_2 \frac{w}{w_c}} \tag{2}$$

$$\hat{\sigma} = 1 - \hat{w} \tag{3}$$

$$\hat{\sigma} = \begin{cases} 1 - \left(1 - \hat{f}_1\right)\dfrac{\hat{w}}{\hat{w}_1}, & 0 \le \hat{\sigma} \le \hat{f}_1 \\ \dfrac{\hat{f}_1}{1 - \hat{w}_1}(1 - \hat{w}), & \hat{f}_1 < \hat{\sigma} \le 1 \end{cases} \tag{4}$$

$$\hat{\sigma}^m + \hat{w}^{2m} = 1 \quad \text{or} \quad \hat{\sigma} = 1 - \hat{w}^{2n} \tag{5}$$

In (2)-(5), C_1, C_2, m, n are material constants, $\hat{\sigma} = \sigma / f_t$, $\hat{w} = w / w_c$, $\hat{f}_1 = f_1 / f_t$, and $\hat{w}_1 = w_1 / w_c$, where w_1 and f_1 are the co-ordinates of the knee of the bilinear law. Xiao & Karihaloo (2006) used the eigenfunction expansion method of Williams (1957) and combined it with the complex function formalism of Muskhelishvili (1953) in the spirit of Sih and Liebowitz (1968) to solve the fields. The cohesive and frictional laws on the crack faces were imposed through appropriate boundary conditions. For a general plane mixed mode I + II problem, the complex functions $\phi(z)$ and $\chi(z)$ were chosen as series of complex eigenvalue Goursat functions

$$\phi(z) = \sum_{n=0}^{\infty} \left(a_n^1 + i a_n^2\right) z^{\lambda_n}, \qquad \chi(z) = \sum_{n=0}^{\infty} \left(b_n^1 + i b_n^2\right) z^{\lambda_n + 1}$$

Below we will use the designations $a_{1n} = a_n^1$, $a_{2n} = a_n^2$, $b_{1n} = b_n^1$ and $b_{2n} = b_n^2$.

To complete the asymptotic analysis of the crack tip fields, solutions need to satisfy the proper symmetry conditions along the line of extension of the cohesive crack, and boundary conditions on the cohesive crack faces.

If normal cohesive separation applies to the crack faces, relationship (1) needs to be satisfied over the cohesive zone. The stresses at the cohesive crack tip are non-singular (because the SIF K_I = 0). Moreover, the following conditions need to be satisfied:

(a) if the cohesive crack faces are frictionless

$$\sigma_y\big|_{\theta=\pi} = \sigma_y\big|_{\theta=-\pi} \neq 0, \quad \tau_{xy}\big|_{\theta=\pi} = \tau_{xy}\big|_{\theta=-\pi} = 0 \tag{6}$$

(b) if the Coulomb friction is considered

$$\sigma_y\big|_{\theta=\pi} = \sigma_y\big|_{\theta=-\pi} \neq 0, \quad \tau_{xy}\big|_{\theta=\pi} = \tau_{xy}\big|_{\theta=-\pi} = -\mu_f \sigma_y\big|_{\theta=\pm\pi} \neq 0 \tag{7}$$

where μ_f equals the positive or negative value of the coefficient of kinetic friction, which is assumed to be constant, depending on the relative sliding direction of the two crack faces. Specifically, $\mu_f > 0$ when $\delta > 0$ and $\mu_f < 0$ when $\delta < 0$.

(c) if the cohesive crack faces are in pure mode I condition

$$\sigma_y\big|_{\theta=\pi} = \sigma_y\big|_{\theta=-\pi} \neq 0, \quad \tau_{xy}\big|_{\theta=\pi} = \tau_{xy}\big|_{\theta=-\pi} = 0, \quad \tau_{xy}\big|_{\theta=0} = 0, \text{ and } v\big|_{\theta=0} = 0 \tag{8}$$

In all three situations, the length of the process (cohesive) zone is either prescribed (i.e. an initial cohesive zone exists before the loading is applied, and does not propagate under the present loading) or is determined by the condition $w = w_c$ in the normal cohesion-separation relation (1) at the instant of growth of the pre-existing traction-free crack.

Frictionless Cohesive Crack with Normal Cohesive Separation

The relationship (1) between cohesion and normal separation will be discussed below. After considering conditions (6) on the crack faces, the solutions are composed of two parts. The first part corresponds to integer eigenvalues

(a) $\lambda_n = n + 1$, $b_{2n} = -\dfrac{n}{n+2} a_{2n}$, $n = 0, 1, 2, \cdots$, (9)

giving

$$\hat{\sigma}_y = \frac{\sigma_y\big|_{\theta=\pm\pi}}{f_t} = \sum_{n=0} c_n r^n = 1 + \sum_{n=1} c_n r^n \tag{10}$$

where

$$c_n = \frac{(n+2)(n+1)(a_{1n}+b_{1n})\cos n\pi}{f_t}, \quad c_0 = \frac{2(a_{10}+b_{10})}{f_t} = 1 \tag{11}$$

since $\sigma_y\big|_{\theta=\pm\pi} = f_t$ when $r \to 0$.

The opening and sliding displacements of the cohesive crack faces vanish for integer eigenvalues

$$w = 0, \quad \delta = 0 \tag{12}$$

The second part of the asymptotic solutions corresponds to non-integer eigenvalues

$$\text{(b)} \ \lambda_n = \frac{2n+3}{2}, \ b_{1n} = -\frac{2n+1}{2n+5} a_{1n}, \ b_{2n} = -a_{2n}, \ n = 0, 1, 2, \cdots, \tag{13}$$

giving

$$\sigma_y\big|_{\theta=\pm\pi} = 0 \tag{14}$$

$$\hat{w} = \frac{w}{w_c} = \sum_{n=0} \bar{d}_n r^{\frac{2n+3}{2}}, \ \bar{d}_n = \frac{\left[\left(\kappa + \frac{2n+3}{2}\right) a_{1n} + \frac{2n+5}{2} b_{1n}\right] \sin\frac{2n+3}{2}\pi}{\mu w_c} \tag{15}$$

$$\delta = \sum_{n=0} \frac{r^{\frac{2n+3}{2}}}{\mu} \left[\left(\frac{2n+3}{2} - \kappa\right) a_{2n} + \frac{2n+5}{2} b_{2n}\right] \sin\frac{2n+3}{2}\pi \tag{16}$$

Consider the truncated $N + 1$ terms of $\hat{w}$ (15), and denote $d_0 = \bar{d}_0$, $d_n = \bar{d}_n / d_0$ $(n > 1)$

$$\hat{w} = d_0 r^{\frac{3}{2}} \left(1 + \sum_{n=1}^{N} d_n r^n\right) \tag{17}$$

The expansion of $\hat{w}$ raised to the power $2i/3$ is also truncated to $N + 1$ terms, since these terms include only the truncated $N + 1$ terms of $\hat{w}$. Hence

$$\hat{w}^{\frac{2}{3}i} = d_0^{\frac{2}{3}i} r^i \left(1 + \sum_{n=1}^{N} \beta_{in} r^n\right) \tag{18}$$

where

$$\beta_{in} = \frac{f_i^{(n)}(0)}{n!}, \ f_i(r) = \left(1 + \sum_{n=1}^{N} d_n r^n\right)^{\frac{2}{3}i}, \tag{19}$$

where $f_i^{(n)}(0)$ denotes the nth derivative at $r = 0$.

The derivatives of function $f_i(r)$ (19) are given in Xiao & Karihaloo (2006) and the corresponding first five coefficients β_{in} are

$$\begin{aligned}
\beta_{i1} &= \frac{2}{3} i d_1 \\
\beta_{i2} &= \frac{1}{3} i \left(\frac{2}{3}i - 1\right) d_1^2 + \frac{2}{3} i d_2 \\
\beta_{i3} &= \frac{1}{9} i \left(\frac{2}{3}i - 1\right)\left(\frac{2}{3}i - 2\right) d_1^3 + \frac{2}{3} i \left(\frac{2}{3}i - 1\right) d_1 d_2 + \frac{2}{3} i d_3 \\
\beta_{i4} &= \frac{1}{36} i \left(\frac{2}{3}i - 1\right)\left(\frac{2}{3}i - 2\right)\left(\frac{2}{3}i - 3\right) d_1^4 + \frac{1}{3} i \left(\frac{2}{3}i - 1\right)\left(\frac{2}{3}i - 2\right) d_1^2 d_2 + \frac{1}{3} i \left(\frac{2}{3}i - 1\right) d_2^2 + \frac{2}{3} i \left(\frac{2}{3}i - 1\right) d_1 d_3 + \frac{2}{3} i d_4 \\
\beta_{i5} &= \frac{1}{180} i \left(\frac{2}{3}i - 1\right)\left(\frac{2}{3}i - 2\right)\left(\frac{2}{3}i - 3\right)\left(\frac{2}{3}i - 4\right) d_1^5 + \frac{1}{9} i \left(\frac{2}{3}i - 1\right)\left(\frac{2}{3}i - 2\right)\left(\frac{2}{3}i - 3\right) d_1^3 d_2 + \frac{1}{3} i \left(\frac{2}{3}i - 1\right)\left(\frac{2}{3}i - 2\right) d_1 d_2^2
\end{aligned} \tag{20}$$

$$+\frac{1}{3}i\left(\frac{2}{3}i-1\right)\left(\frac{2}{3}i-2\right)d_1^2 d_3+\frac{2}{3}i\left(\frac{2}{3}i-1\right)d_2 d_3+\frac{2}{3}i\left(\frac{2}{3}i-1\right)d_1 d_4+\frac{2}{3}id_5$$

With the use of (18), if we choose $N = 5$, then after satisfying the cohesive relationship (1) we can obtain the following expressions for coefficients c_n appearing in (10) (see Xiao & Karihaloo, 2006):

$$\begin{aligned}
c_1 &= \alpha_1 d_0^{\frac{2}{3}} \\
c_2 &= \alpha_2 d_0^{\frac{4}{3}} + \alpha_1 d_0^{\frac{2}{3}}\beta_{11} \\
c_3 &= \alpha_3 d_0^2 + \alpha_1 d_0^{\frac{2}{3}}\beta_{12} + \alpha_2 d_0^{\frac{4}{3}}\beta_{21} \\
c_4 &= \alpha_4 d_0^{\frac{8}{3}} + \alpha_1 d_0^{\frac{2}{3}}\beta_{13} + \alpha_2 d_0^{\frac{4}{3}}\beta_{22} + \alpha_3 d_0^2\beta_{31} \\
c_5 &= \alpha_5 d_0^{\frac{10}{3}} + \alpha_1 d_0^{\frac{2}{3}}\beta_{14} + \alpha_2 d_0^{\frac{4}{3}}\beta_{23} + \alpha_3 d_0^2\beta_{32} + \alpha_4 d_0^{\frac{8}{3}}\beta_{41} \\
c_6 &= \alpha_1 d_0^{\frac{2}{3}}\beta_{15} + \alpha_2 d_0^{\frac{4}{3}}\beta_{24} + \alpha_3 d_0^2\beta_{33} + \alpha_4 d_0^{\frac{8}{3}}\beta_{42} + \alpha_5 d_0^{\frac{10}{3}}\beta_{51} - \left(1+\sum_{i=1}^{5}\alpha_i\right)d_0^4 \\
c_7 &= \alpha_2 d_0^{\frac{4}{3}}\beta_{25} + \alpha_3 d_0^2\beta_{34} + \alpha_4 d_0^{\frac{8}{3}}\beta_{43} + \alpha_5 d_0^{\frac{10}{3}}\beta_{52} - \left(1+\sum_{i=1}^{5}\alpha_i\right)d_0^4\beta_{61} \\
c_8 &= \alpha_3 d_0^2\beta_{35} + \alpha_4 d_0^{\frac{8}{3}}\beta_{44} + \alpha_5 d_0^{\frac{10}{3}}\beta_{53} - \left(1+\sum_{i=1}^{5}\alpha_i\right)d_0^4\beta_{62} \\
c_9 &= \alpha_4 d_0^{\frac{8}{3}}\beta_{45} + \alpha_5 d_0^{\frac{10}{3}}\beta_{54} - \left(1+\sum_{i=1}^{5}\alpha_i\right)d_0^4\beta_{63} \\
c_{10} &= \alpha_5 d_0^{\frac{10}{3}}\beta_{55} - \left(1+\sum_{i=1}^{5}\alpha_i\right)d_0^4\beta_{64} \\
c_{11} &= -\left(1+\sum_{i=1}^{5}\alpha_i\right)d_0^4\beta_{65}
\end{aligned} \tag{21}$$

Note that the above asymptotic solution is not for pure mode I cohesive crack tip (cf. (6) and (8)), since along the line of extension of the crack, $\theta = 0$, the shear stress does not vanish ($\tau_{xy} \neq 0$).

For non-integer eigenvalues (13), the coefficients a_{1n} and a_{2n} may be regarded as independent, so that coefficients b_{1n} are linearly dependent on a_{1n} and b_{2n} on a_{2n}. For integer eigenvalues (9), coefficients a_{1n} and a_{2n} may also be regarded as independent, so that coefficients b_{2n} now depend linearly on a_{2n}. However, the coefficients b_{1n} for integer eigenvalues will depend both linearly on a_{1n} for integer eigenvalues and nonlinearly on a_{1n} for non-integer eigenvalues via (15), (16), (20), and (21). The inherent nonlinear nature of the problem is reflected in these nonlinear relationships between the coefficients of the asymptotic fields.

The displacements corresponding to $\lambda_{-1} = 0$, or $n = -1$ in (9) are rigid body translations at the crack tip

$$2\mu u_{-1} = \kappa a_{1,-1} - b_{1,-1}, \quad 2\mu v_{-1} = \kappa a_{2,-1} + b_{2,-1} \tag{22}$$

The displacements corresponding to a_{20} ($n = 0$, $\lambda_0 = 1$ and $b_{20} = 0$ from (9)) represents rigid body rotation with respect to the crack tip

$$2\mu\hat{u}_0 = -r(\kappa+1)a_{20}\sin\theta, \quad 2\mu\hat{v}_0 = r(\kappa+1)a_{20}\cos\theta \tag{23}$$

Coulomb Frictional Cohesive Crack with Normal Cohesive Separation

In principle, a cohesive relationship can also be considered in the tangential direction. However, this is a contentious issue, since it is difficult to separate the cohesive-sliding relation from the frictional force between the rough cohesive crack faces. Hence we will only consider the Coulomb friction between the crack faces instead of a tangential cohesive relationship. The corresponding boundary conditions are (7).

The complete asymptotic solutions are again composed of two parts. The first part corresponding to integer eigenvalues is similar to that above but with different constraints on the coefficients

$$\lambda_n = n+1,\ na_{2n} + (n+2)b_{2n} = -\mu_f(n+2)(a_{1n}+b_{1n}),\ n = 0, 1, 2, \cdots \tag{24}$$

From (24), we have

$$b_{2n} = -\frac{n}{n+2}a_{2n} - \mu_f(a_{1n}+b_{1n})$$

When $\mu_f = 0$, the cohesive crack faces are frictionless, and (24) reduces to (9). These solutions have non-zero σ_y and τ_{xy} along the cohesive crack faces, but zero crack opening w and sliding δ. The second part of the asymptotic solutions corresponding to non-integer eigenvalues satisfies

$$b_{1n} = -\frac{\lambda_n - 1}{\lambda_n + 1} a_{1n},\ b_{2n} = -a_{2n},\ (\mu_f a_{1n} - a_{2n})\cos(\lambda_n - 1)\pi = 0 \tag{25}$$

Assuming that

$$\mu_f a_{1n} - a_{2n} \neq 0 \tag{26}$$

the third equation in (25) gives

$$\cos(\lambda_n - 1)\pi = 0 \tag{27}$$

so that the second part of asymptotic solutions is identical to (14) – (16).
The remaining solution procedure and final asymptotic solutions as well as the dependence of the coefficients are similar to those for frictionless cohesive cracks. Equations (22) and (23) again represent the rigid body modes for the present case.
Finally, we give without detail the leading term in the asymptotic displacement field of a pure mode I cohesive crack (with the boundary conditions (8)). This term is given by the lowest non-integer eigenvalue

$$u = \frac{r^{3/2}}{2\mu} a_{11}\left[\left(\kappa + \frac{1}{2}\right)\cos\frac{3}{2}\theta - \frac{3}{2}\cos\frac{1}{2}\theta\right],\ v = \frac{r^{3/2}}{2\mu} a_{11}\left[\left(\kappa - \frac{1}{2}\right)\sin\frac{3}{2}\theta - \frac{3}{2}\sin\frac{1}{2}\theta\right] \tag{28}$$

where the coefficient a_{11} depends on the geometry and loading of the structure and on the cohesion-separation law.

Implementation of the Asymptotic Fields in XFEM/GFEM

In the implementation of the cohesive crack asymptotic fields as enrichment functions in the XFEM/GFEM, if not only the first term but also the higher order terms are used as in Liu et al. (2004), the linear dependence of the coefficients can be enforced in advance, while the nonlinear dependence of the coefficients can be enforced as constraints in the solution process. It is more convenient to use only the leading term of the displacement asymptotic field at the tip of a cohesive crack (which ensures a displacement discontinuity normal to the cohesive crack face) as the enrichment function, as in most implementations of the XFEM in the literature. The complete implementation with several examples can be found in Xiao et al. (2007).

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