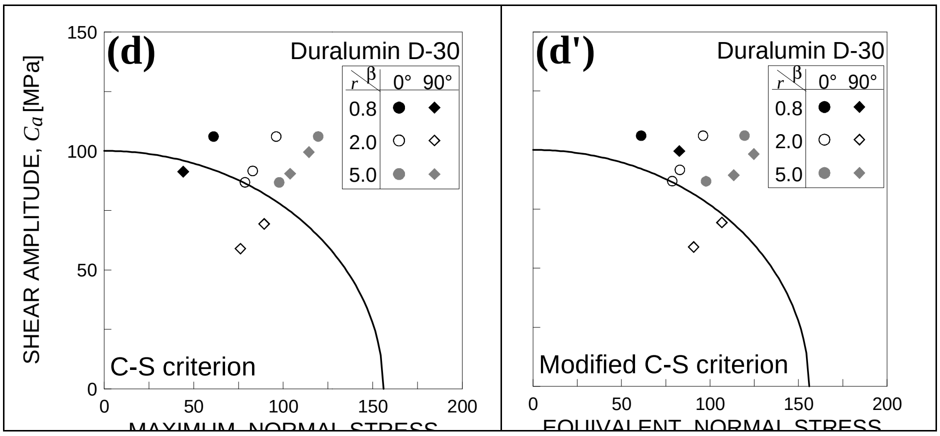


**Figure 2** Shear stress amplitude against maximum normal stress (diagrams (a) to (d)) and equivalent normal stress amplitude (diagrams (a') to (d')) acting on the critical plane: theoretical evaluations according to both the original or modified C-S criterion, and experimental results [20].

## ORDERED CRACK SYSTEM FORMATION AT THERMAL SHOCK


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### Introduction

Fracture of brittle materials by cracks formation and growth under thermal action is intensively studied (see, e.g. [1-6]). The problem is interesting, in particular, because of often observed damage and fracture of structural elements fabricated from ceramic materials and glasses at operation under an extremal thermomechanical loading. Multiple fracture is usually observed. A hierarchical ordered crack system (structure of cracks) in the form of a pattern or system of subparallel cracks of different sizes occurs in a loaded body (often on its surface). As a result a body region is unloaded from excessive thermoelastic stresses.

An analysis of thermomechanical fracture is aimed at searching for the conditions of initiation and growth of separate cracks, sequence of events leading to cracks system occurring and growth, conditions of cracks front stability, residual strength and longevity of damaged components, as well as, the methods for determining the thermal strength of materials and components under various loading history. A theoretical analysis of the appropriate problems of solid mechanics for bodies with cracks is rather complex. That is why as a rule the examples of simple stress distributions in the bodies of simple geometry are considered. Simplifying assumptions and approximate methods are used for solving the problems. In particular, this is related to experimental methods which one uses to determine thermostrength properties of materials and components. Majority of the methods are based on evaluation of the thermal stress level by the following semi-emperical formula

$$\sigma = \frac{\alpha E \Delta T K_\phi}{(1-\mu)} \qquad (1)$$

where $\alpha$ is the thermal expansion coefficient, $E$ is the Young modulus, $\mu$ is the Poisson ratio, $\Delta T$ is the temperature drop between the average body temperature and temperature in the region where the stresses are estimated, $K_\phi$ is the shape coefficient.

In this connection asymptotic methods, in which characteristic features of multiple fracture process are used, seem to be interesting. One of such methods is suggested in the given paper. The method is based on the evident effect inherent to the fracture process accompanied by formation of a system of subparallel cracks of different sizes. Indeed, only largest cracks growth is adjusted by the stress field in the scale of the whole body with these cracks. For other cracks in the hierarchy the following effect takes place: the less are the crack length and the distance between them the less is the scale of local stress field having an influence on their growth. In other words, if an advanced process of multiple fracture occurs in a region damaged by initial cracks then thermomechanical processes in separate strips (beams) separated by initial cracks became to be independent on the situation for the whole body. A temperature distribution in the body is not influenced by presence of cracks if a heat exchange on the crack surfaces is small (this is typical for practical problems because of a small opening of the brittle fracture cracks.)

Hence, the following approximate approach to an analysis of an advanced stage of multiple fracture at intensive thermal loads is possible: the problem of the heat transfer is solved for the whole body while the crack growth is considered within the isolated beams separated in the body by parallel or subparallel cracks on the preceding stage of fracture. In this case the process of the crack growth is assumed to be isothermal one. This approach is used latter on for solving some model problems.

**1. A beam model of thermal fracture**

Let us consider a stress state of a thin beam-half-strip under the action of sharp cooling on the free edge at absence of heat exchange on lateral sides. As it was pointed out in the Introduction, this situation is related to a thermal shock at the boundary of a half-plane separated by parallel cracks oriented transverse to the boundary on a series of beams. Let us pay attention to existence of a simple asymptotics of the stress state for a thin beam. Indeed, if a beam is very thin (its width tends to zero) then thermal deformations in transverse direction do not lead to occurring meaningful stresses in this direction ($\sigma_y$ ~ 0). So, one can neglect by the stresses $\sigma_y$ if the temperature gradients along the beam length are small (temperature through the beam section is assumed to be constant). Now let us increase the beam width. For this aim we combine it from two half-beams of small width with the preceding thermal action. The condition of deformations compatibility requires that the lateral beams surfaces were tight. Thermal deformations, $\varepsilon_y$, in each beam pevent this. Note, that deformations in the adjacent half-beams are symmetric relative to the median contact line at the accepted conditions. Compensation of displacements caused by these deformations leads to occurring transverse stresses. In the case of two half-beams (Fig. 1) one can write

$$\Delta u = \frac{\tau \Delta x}{G} \tag{2}$$

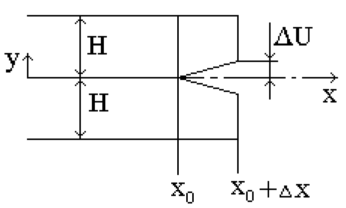



Fig. 1

where $\Delta u \sim (\Delta\varepsilon)H$, $\Delta\varepsilon$ is the deformation increment on the interval $\Delta x$, $H$ is the thickness of a half-beam, $\tau$ is the average stress in the section, $G$ is the shear modulus.

After limit transition at $(\Delta x)\to 0$ and changing $G$ by $E$ we obtain

$$\sigma_y(x) \approx \frac{EH}{2(1+\mu)}\frac{d\varepsilon}{dx}; \quad \text{where} \quad \frac{d\varepsilon}{dx} = \frac{dT}{dx}\alpha \tag{3}$$

Remind for comparison the formula for quasi-static thermal stresses $\sigma_y$ in a half-space [7]

$$\sigma_y = \frac{E\alpha T(x)}{(1-\mu)} \tag{4}$$

As an example let us consider a thermal shock at the edge of a half-plane separated on the beams of width 2H. The temperature field in case of a jump-type change of the surface temperature (the coefficient of heat exchange h→∞) is determined by the following formula [7]

$$T = \Delta T \ \mathrm{erfc}\left(\frac{x}{2\sqrt{at}}\right) \tag{5}$$

where $\alpha$ is the coefficient of thermal conductivity, $\Delta T$ is the temperature drop between the surface and a half-plane point remote from the surface. Then

$$\frac{d\varepsilon}{dx} = -\frac{\alpha\Delta T}{\sqrt{\pi at}} e^{-\frac{x^2}{4at}} \tag{6}$$

According to Eqs (3) and (6)

$$\sigma_y(x,t) = -\frac{EH}{2(1+\mu)}\frac{\alpha\Delta T}{\sqrt{\pi at}} e^{-\frac{x^2}{4at}} \tag{7}$$

The stress variation in a thin beam is given in Fig. 2. In the same figure the quasistatic solution (4)

$$\sigma_y(x,t) = -\frac{E\alpha\Delta T}{1-\mu}\left(\mathrm{erfc}\frac{x}{2\sqrt{at}}\right) \tag{8}$$

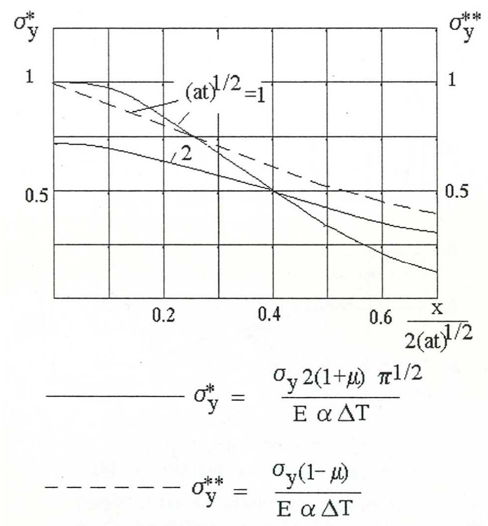



Fig. 2

is also presented for comparison. One can see that the stress variation in the beam is more gently sloping. The crack growth character at further fracture is influenced by this. The stresses $\sigma_y(x,t)$ decrease monotonically with the depth of the body. The maximum stress level at the free edge decreases and the size of the stressed region increases with increasing exposition time. Such stress picture corresponds to the known scheme of fracture processes [1-3] at the edge of a half-plane. At the first stage a set of short cracks is initiated. These cracks grow with the movement of the stress front in depth of the body. A part of them is arrested such that the amount of moving cracks decreases with their length increasing. Conditions of cracks retardation (or the conditions of the cracks front stability) are discussed, e.g. in [2, 3].

Let us show that the described above approach makes possible another asymptotic analysis of the multiple fracture. An elementary cell of the crack system at the advanced stage of the process can be separated as a beam with lateral sides being surfaces of subparallel cracks while a crack of smaller length grows along the middle plane of the beam. Then the growth of the system of parallel cracks can be considered as the growth of individual cracks each of which propagates in the middle plane of its effective beam.

Consider the conditions of the crack growth along a beam loaded by thermoelastic stresses given by Eq. (7) for the case $h \to \infty$. Since the parallel cracks simultaneously grow in adjacent beams one can see from the conditions of deformations compatability that a neutral line, where transverse displacements are absent, exists between them. Let us identify this line location for each pair of adjacent cracks with the lateral sides of effective beams. Thus, the following scheme can be used for evaluation of a crack limit equilibrium. A beam-half-strip with a crack on the middle plane is loaded by the stresses $\sigma_y(x,t)$ symmetrically on its surfaces while transverse displacements on the beam lateral sides are equal to zero (Fig. 3).

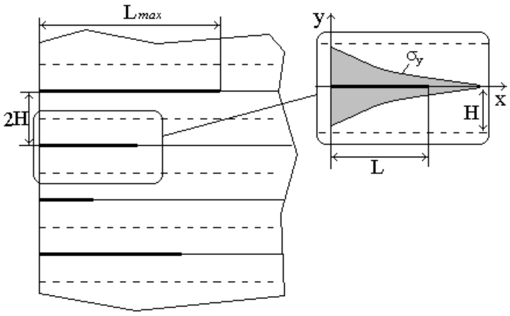



Fig. 3

It is convenient to use the compliance method [8] for estimation of the stress intensity factor $K_I$ for the scheme. The elastic energy stored in a deformed beam of width $2H$ equals

$$U = \int_o^{\ell} \frac{\sigma_y^2(x,t)(1-\mu^2)}{E} H\,dx \qquad (9)$$

According to the compliance method one can write for the symmetric loading

$$\frac{dU}{d\ell} = \frac{K_I^2(1-\mu^2)}{E}; \quad K_{II} = K_{III} = 0 \qquad (10)$$

Hence,

$$K_I = \sigma_y(x,t)\sqrt{H} \qquad (11)$$

Note, that this formula coincides by form with one given in [9] for the constant load $\sigma_y = q = \text{const}$. By incorporating Eq. (7) we obtain

$$K_I = A\frac{H^{3/2}}{\sqrt{t}} e^{\frac{x^2}{4at}} \quad \text{where } A = \frac{E\alpha\Delta T}{2(1+\mu)\sqrt{\pi a}} \qquad (12)$$

One can see from Eq. (12) that $K_I$ is nonmonotone function of time ($K_I$ equals zero at $t = 0$ and $t \to \infty$).

Let us accept the evident assumption: crack growth occurs in a regime providing the maximal value of the stress intensity factor in its tip, i.e. in the regime being most favorable for dissipation of the beam elastic energy. Then

$$\frac{dK_I}{dt} = AH^{3/2} e^{-\frac{x^2}{4at}} \left( \frac{x^2}{4at^{3/2}} - \frac{1}{2t^{3/2}} \right) = 0 \qquad (13)$$
$$t = \frac{x^2}{2a}$$

and

$$K_{Imax} = \frac{AH^{3/2}}{\sqrt{e}} \frac{\sqrt{2a}}{x} \qquad (14)$$

Denote by $x = \ell$ the crack length in the conditions of the limit equilibrium

$$K_{Imax} = K_{Ic} \qquad (15)$$

where $K_{Ic}$ is the critical stress intensity factor. Eqs (14), (15) lead to the interrelation between the effective beam width, $H$, and crack length

$$\ell = \frac{AH^{3/2}}{K_{Ic}} \frac{\sqrt{2a}}{e} \qquad (16)$$

This simple relation, obtained for conditions of growth of a single crack within the series of parallel ones, enables to perform some estimates for the whole cracks ensemble. For this aim it is sufficient to assume that a beam of larger effective width can be combined from several beams of smaller widths. Of course the estimates of the crack concentration (density) will be upper estimates since a part of elastic energy is released in vicinities of more large cracks on formation and growth the smaller ones. As an integral characteristic of a crack system in this approximation we will use an interrelation between the relative crack density (the ratio of amount of cracks with the length exceeding a certain value, $\ell$, to the total amount of cracks) and the relative crack length (the ratio of the size, $\ell$, to the maximal crack length, $\ell_{max}$), in the system. Such form of representation was given, in particular, in [3].

According to set forth the relative crack density can be written as follows

$$\frac{n(x>\ell)}{n_\Sigma} \approx \frac{\int_{\ell}^{\ell_{max}} H(\ell)^{-1} d\ell}{\int_{o}^{\ell_{max}} H(\ell)^{-1} d\ell} \quad (17)$$

By incorporating Eq. (16) we obtain from Eq. (17)

$$\frac{n(x>\ell)}{n_\Sigma} \approx 1 - \left(\frac{\ell}{\ell_{\max}}\right)^{1/3} \quad (18)$$

This result is given in Fig. 4. For the sake of comparison on the same figure are presented the data obtained in [3] on the basis of numerical solution of the full elasticity problem on many cracks at the half-plane edge as well as the experimental data given in [10] for similar loading variant. One can see that the relation (18) in fact represents the estimate of the relative crack density in a large part of the crack length range. Note also, that thermal properties and other material parameters are accounted for in Eq. (18) only through the temperature distribution. Hence, the dependence (18) is universal in that sense.

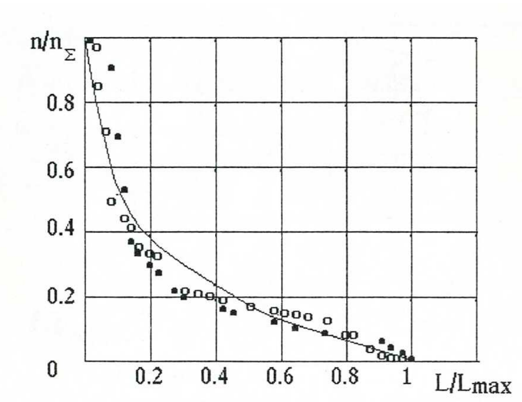



Fig. 4

Other forms of universal interrelations which give approximate description of an integral parameter of an hierarchical crack system at a thermal shock can be used in practical problems. For instance, one can compare the density of cracks with the length less than a certain size, $\ell$ , and the total amount of cracks. It follows from Eq. (18) that

$$\frac{n(x>\ell)}{n_\Sigma} \approx \left(\frac{\ell}{\ell_{\max}}\right)^{1/3} \quad (19)$$

The cracks of characteristic sizes, e.g. minimal and maximal cracks in the system can be compared also. By incorporating Eq. (16) we obtain that the ratio of distances between the adjacent maximal and minimal cracks equals

$$\frac{H_{\max}}{H_{\min}} \approx \left(\frac{\ell_{\max}}{\ell_{\min}}\right)^{2/3} \quad (20)$$

This relation can be used at evaluation of the maximal crack length having the results of measurement or evaluation of a distance between the small cracks. Assume, for instance, that $H_{\max}$ is of order of the body size, $L$. Estimate the value $\ell_{\min}$ from the condition of local strength as $\ell_{\min} \sim \left(K_{\mathrm{Ic}}/\sigma^*\right)^2$ where $\sigma^*$ is characteristic local strength. Then we obtain from Eq. (20)

$$\ell_{\max} \sim \left(\frac{K_{\mathrm{Ic}}}{\sigma^*}\right)^2 \left(\frac{L}{H_{\min}}\right)^{3/2} \quad (21)$$

Note, that transformations performed with Eqs (7) and (12)-(16) are in fact equivalent to construction of an envelope, i.e. construction of a function of one variable, x, by elimination of the parameter, $t$, considering the function of two variables. It is easily to see that Eq. (16) can be obtained if in Eq. (12) instead of the function $\sigma_y(x,t)$ one will use the envelope $\sigma_y(x)$ given by Eq. (7)

$$\sigma_y(x) = -A\sqrt{\frac{2a}{e}}\frac{H}{x} \quad (22)$$

Hence, one can conclude that an approximate quasi-static analysis of multiple fracture at a thermal shock can be performed within the model of crack system growth in time, in which the time parameter is eliminated, while all cracks are considered being in the state of the limit equilibrium under the action of stresses given by the envelope of the stress function in time. This approach can be extended for other variants of loading.

## 2. Estimates for finite bodies of simple shapes

As an example let us consider a scheme of disk fracture in case of the heat flux directed to the external cylindrical surface while the edge sections of the disk are heat insulated. The problem is related to thermal fracture of high strength ceramics (e.g. [4-6]).

For the sake of convenience assume that the temperature distribution along the disk radius is quasi-steady and is described by parabolic or logarithmic functions [5]

$$T(r) = T_{in} + \Delta T\left(\frac{r}{R}\right)^n; \quad n \sim 2 \quad (23)$$

$$T(r) = T_{ou} + \Delta T \ln\left(\frac{r}{R}\right) \quad (24)$$

where R is the disk radius, $T_{in}$ and $T_{ou}$ are the temperature of the disk center and its side surface, respectively.

Quasi-steady distributions (23) and (24) can be considered as different envelopes of a nonstationary distribution and will be used later on in that sense.

A system of radial cracks separating, according to the scheme given in Sect. 1, a set of wedgelike beams is formed at multiple fracture caused by tensile stresses at the disk surface.

Assume that the separated beams are thin. Then according to Eq. (2) we can write for a thin wedge shaped beam

$$\sigma_\theta = \frac{EH(r)\alpha}{2(1+\mu)}\frac{dT}{dr}; \quad \sigma_r \approx \sigma_\theta \frac{H_o}{R}; \quad H(r) = H_o \frac{r}{R} \quad (25)$$

where $H_o$ is the beam width on the disk surface.

By incorporating Eqs (23)-(25) we obtain the stresses $\sigma_\theta$ for the parabolic distribution

$$\sigma_\theta = \frac{E\Delta T H_o \alpha n}{2(1+\mu)R}\left(\frac{r}{R}\right)^n \quad (26)$$

and for the logarithmic distribution

$$\sigma_\theta = \frac{E\Delta T H_o \alpha}{2(1+\mu)R} \quad (27)$$

Note, that the structure of Eq. (27) is close to semi-empirical relation (1) being used in experimental practice of thermomechanical properties determination.

Let the origin of coordinates is located in the radial crack mouth on the outer disk generator. Then the condition of the limit equilibrium of the central radial crack in thin narrowing beam has the form (according to Eqs (11) and (15))

$$K_{\mathrm{Ic}} = \sigma_\theta \sqrt{H} \quad (28)$$

Hence, we obtain the following interrelation between $H_o$ and the length of the crack being in the limit equilibrium, $\ell$,

$$H_o = \left( \frac{K_{Ic} R}{nB\left(1-\frac{\ell}{R}\right)^{n+1/2}} \right)^{2/3} \quad (29)$$

for the parabolic distribution and

$$H_o = \left(\frac{K_{Ic}R}{B}\right)^{2/3}\left(1-\frac{e}{R}\right)^{-1/3} \qquad B = \frac{E\alpha\Delta T}{2(1+\mu)} \quad (30)$$

for the logarithmic one.

The interrelation between the relative crack density and their relative length (similar to Eq. (17)) can be obtained from Eqs (28), (29)

$$\frac{n(x>\ell)}{n_\Sigma} \approx \frac{\int_{\ell/R}^{1} H_o^{-1} d\left(\frac{\ell}{R}\right)}{\int_{o}^{1} H_o^{-1} d\left(\frac{\ell}{R}\right)} \quad (31)$$

Finally, for the parabolic distribution

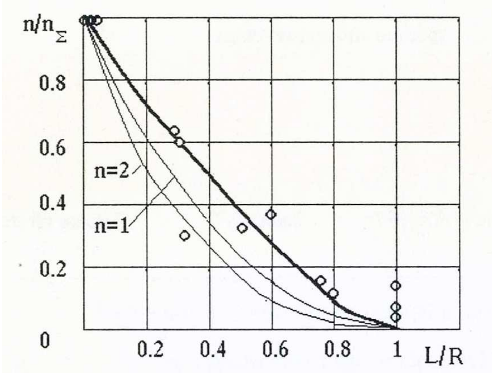
$$\frac{n(x>\ell)}{n_\Sigma} \approx \left(1-\frac{\ell}{R}\right)^{8/3} \quad \text{at} \quad n=2$$

$$\frac{n(x>\ell)}{n_\Sigma} \approx \left(1-\frac{\ell}{R}\right)^{2} \quad \text{at} \quad n=1 \quad (32)$$

and for the logarithmic one

$$\frac{n(x>\ell)}{n_\Sigma} \approx \left(1-\frac{\ell}{R}\right)^{4/3} \quad (33)$$

The appropriate curves are given in Fig. 5 where the experimental results for thin disks of infusible ceramic material (Zirconium carbide) obtained in [4] are also shown. One can see, that the logarithmic temperature distribution provides better correlation with the experimental data as compared to the parabolic distribution. Perhaps, this is related to the conditions of heat-transfer at the outer cooled surface of the disks.




results given by Eq. (33)
results given by Eq. (32)
o experimental results for ZrC disks (diameter ~ 5mm, thickness 1.5 – 2mm) [4]

Fig.5

## 3. To an estimate of the residual strength of a body with a system of many cracks

Abrupt cooling of the disks lateral surface is accompanied by growth of many radial cracks from the surface to the disk center, while abrupt heating at the lateral surface leads to initiation and growth of many cracks in from the central part of the disk. Experiments show that in the second case full disk fracture (its separation on parts) occurs more often while in the first case full fracture is a rare phenomenon.

Let us show within the framework of the beam approximation, that this observed effect can be related to the conditions of growth of a maximal size crack. First, let us find the value of the stress intensity factor $K_{\mathrm{I}}$ and their derivatives relative to the crack length for the cracks of maximal size $\ell \to R$. Consider, as an example, the case of the parabolic temperature distribution along the disk.

For the cracks growing from the disk lateral surface we obtain using Eqs (26), (28)

$$K_I = \frac{H_o^{3/2}Bn}{R}\left(1-\frac{\ell}{R}\right)^{n+1/2} \quad (34)$$

$$\frac{dK_I}{d(\ell/R)} = -\frac{H_o^{3/2}Bn(n+1/2)}{R}\left(1-\frac{\ell}{R}\right)^{n-1/2} \quad (35)$$

such that $K_{\mathrm{I}} \to 0$ and $dK_{\mathrm{I}}/d(\ell/R) \to 0$ at $\ell \to R$; the function $K_{\mathrm{I}}$ monotonically decreases as the crack moves to the central part of the disk.

Now, consider similarly a system of cracks growing from the central part of the disk and analyze the stage of many cracks advance to the disk periphery.

The stresses $\sigma_\theta$ in a wedge shaped beam extending with the crack growth are equal to

$$\sigma_\theta = \frac{H_o Bn}{R}\left(1-\left(\frac{r}{R}\right)^n\right) \qquad (36)$$

where r is referenced from the disk center.

Now, if the crack grows along the wedge shaped beam longitudinal axis from its apex we obtain using Eq. (26)

$$K_I = \frac{H_o^{3/2} Bn}{R}\left(1-\left(\frac{\ell}{R}\right)^n\right)\left(\frac{\ell}{R}\right)^{1/2}$$

$$\frac{dK_I}{d(\ell/R)} = -\frac{H_o^{3/2} Bn}{2R}\left(\frac{R}{\ell}\right)^{1/2}\left(1-2\left(n+\frac{1}{2}\right)\left(\frac{\ell}{R}\right)^n\right) \qquad (37)$$

Again $K_{\mathrm{I}} \to 0$ at $\ell \to R$, while the function $K_{\mathrm{I}}(\ell/R)$ is nonmonotone; the maximal $K_{\mathrm{I}\max}$ is attained at $(\ell/R) = \left[1/2\left(n+(1/2)\right)\right]^{1/n}$, e.g. $(\ell/R)_{K_{I\max}} = (1/5)^{1/2} \approx 0.48$ at $n$ = 2. Hence, the crack growth is unstable up to the length $(\ell/R) \sim 0.48$; its retardation occurs only on the final part of the path, indeed, $dK_I/d(\ell/R) \neq 0$ at $\ell \to R$. The performed analysis confirmers the conclusion [4] (see also [6]) that in case of multiple fracture starting from the disk center (at its abrupt heating on the lateral surface) full fracture of the disk (because of cracks outcome on this surface) is more probable as compared to the case of the abrupt cooling on the lateral surface of the preliminary heated disk.

## Conclusion

Processes of multiple fracture often lead to formation of ordered crack systems (structures of fracture). In particular, such structures are observed in porous and layered materials (media), in zones of non-uniform stress state caused by stress concentrators. Experimental studies and modeling of the processes of ordered fracture in structural and natural materials have a long story. In this connection we would like to mention the review papers [11,12] and recent publications [13-16] where some new results and relevant references are also given.

In the given paper we considered another variant of multiple fracture which is characterized by an ordered crack system formation in the non-uniform stress field caused by a thermal shock. The performed analysis is based on an approach outlined by the authors in the preprint [17]. Note, that an interest to the problems of thermal fracture is strengthened now because of the tendency to create structures operating in the conditions of extremal thermomechanical loading and to develop new technologies providing their high resistance to thermal loading by using special gradient coatings.

## Appendix

The obtained estimates of an influence of adjacent parallel cracks on the stress intensity factor at the tip of a crack of this crack system can be compared with the data given in [18,19]. The stress intensity factors for two and three parallel cracks located transverse to the edge of a strip under tension and bending have been calculated by a numerical method and according to an approximate analytical estimate of the unloading zones near the cracks. The results are presented in the form of the ratio of the calculated values to the strength intensity factor for a single crack under the same loading conditions. A part of these results [19] for the variant of cracks having the same length is given in Fig. 6. It seems that the numerical results for the middle of three cracks of equal length correlates better to the above considered situation of an infinite system of equal parallel cracks. The results of estimates according to the suggested approach (Eq. (18) in the dimensionless form) are also given in the same figure (curve 1). One can see that curve 1 is essentially nearer to the numerical results for three cracks (curve 5) as compared to other estimates which give overestimated values of the stress intensity factor. The difference from the numerical solution does not exceed 10% if the ratio of the distance between the cracks to their length is varied in the range 0.5-1.3.

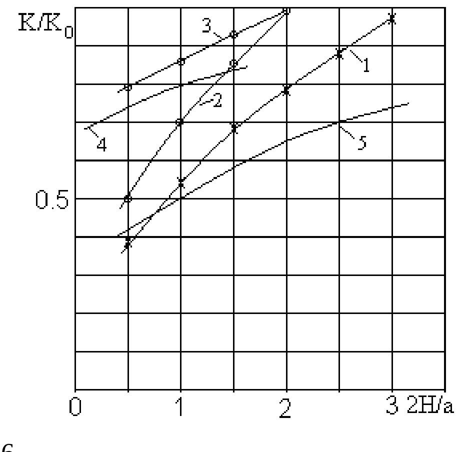



Fig. 6

Different estimates of the adjacent cracks influence on the stress intensity factor for a crack in a system of parallel cracks of equal length, $a$, and distance $2H$ between each other (1 – the obtained estimate for an infinite crack system; 2 – the analytical estimate [19]; 3 – the same for two cracks; 4 – numerical results for two cracks [19]; 5 - the same for three cracks)

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